Lie Symmetry Group of the Nonisospectral Kadomtsev-Petviashvili Equation

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The classical symmetry method and the modified Clarkson and Kruskal (C-K) method are used to obtain the Lie symmetry group of a nonisospectral Kadomtsev-Petviashvili (KP) equation. It is shown that the Lie symmetry group obtained via the traditional Lie approach is only a special case of the symmetry groups obtained by the modified C-K method. The discrete group analysis is given to show the relations between the discrete group and parameters in the ansatz. Furthermore, the expressions of the exact finite transformation of the Lie groups via the modified C-K method are much simpler than those obtained via the standard approach.

Key words: Nonisospectral KP Equation; Classical Symmetry Method; Lie Symmetry Group; Modified C-K Method.

1. Introduction

As it is well known the Kadomtsev-Petviashvili (KP) equation [1] plays an important role in many fields of physics, particularly in fluid mechanics, plasma physics, and gas dynamics. David, Kamran, Levi and Winternitz [2] studied the Lie point symmetry group of the KP equation via the traditional Lie group approach. Lou and Ma [3] developed the direct method presented by Clarkson and Kruskal (C-K) [4] to construct the finite symmetry group of the KP equation. However, recently more and more people study the nonisospectral and variable coefficient generalizations of completely integrable nonlinear evolution equations.

In this paper, we will investigate the nonisospectral KP equation [5, 6]

$$(4u_t + yu_{xxx} + 6yuu_x + 2xu_y)_x + 3yu_{yy} + 4u_y = 0, \quad (1)$$

which may provide more realistic models, in the propagation of (small-amplitude) surface waves in straits or large channels of (slowly) varying depth and nonvanishing vorticity. Here the classical symmetry method and the modified C-K method are used to obtain the Lie symmetry group of this nonisospectral KP equation.

2. Lie Symmetry Group of the Nonisospectral KP Equation Obtained by the Classical Symmetry Method

In this section the classical symmetry method is used to obtain the Lie symmetry group of the nonisospectral KP equation (1). Then some special solutions are given. At first we give a brief outline of the theory of Lie's one-parameter group of transformations for invariance of a partial differential equation with three independent variables [7]. Generalization to more variables is straightforward.

We investigate a general partial differential equation with one dependent variable u and three independent variables x, y, and t:

$$H(x,y,t,u_x,u_y,u_t,u_{xx},u_{xy},u_{xt},u_{yy},u_{yt},u_{tt},\ldots). \quad (2)$$

Let one parameter ε group of transformations of the variables x, y, t and u be taken as

$$\begin{aligned} x' &= f(x,y,t,u;\varepsilon), \quad y' = g(x,y,t,u;\varepsilon), \\ t' &= p(x,y,t,u;\varepsilon), \quad u' = h(x,y,t,u;\varepsilon). \end{aligned} \quad (3)$$

Let $u = \theta(x,y,t)$ be a solution of (2). If we replace the variables u, x, y, t in (2) by $v, x' = f(x,y,t,u;\varepsilon)$, $y' =$

$g(x,y,t,u;\varepsilon)$, $t' = p(x,y,t,u;\varepsilon)$, respectively, (2) becomes

$$H(x',y',t',v,v_{x'},v_{y'},v_{t'},v_{x'x'},v_{x'y'},\\ v_{x't'},v_{y'y'},v_{y't'},v_{t't'}\ldots)=0. \tag{4}$$

Then $v=\theta(x',y',t')$ is a solution of (4). That is to say in the case of transformation (3), $v=\theta(x',y',t')$ is a solution to (4) whenever $u=\theta(x,y,t)$ is a solution to (2). This condition implies if (2) and (4) have a unique solution, then

$$\theta(x',y',t') = h(x,y,t,\theta(x,y,t);\varepsilon). \tag{5}$$

Hence $\theta(x,y,t)$ satisfies the one-parameter function equation

$$\theta(f(x,y,t,\theta;\varepsilon),g(x,y,t,\theta;\varepsilon),p(x,y,t,\theta;\varepsilon))\\ = h(x,y,t,\theta;\varepsilon). \tag{6}$$

Expanding (3) about the identity $\varepsilon=0$, we can generate the following infinitesimal transformations:

$$\begin{aligned} x' &= x+\varepsilon\xi(x,y,t,u)+O(\varepsilon^2),\\ y' &= y+\varepsilon\eta(x,y,t,u)+O(\varepsilon^2),\\ t' &= t+\varepsilon\tau(x,y,t,u)+O(\varepsilon^2),\\ u' &= u+\varepsilon\phi(x,y,t,u)+O(\varepsilon^2). \end{aligned} \tag{7}$$

The functions ξ, η, τ, ϕ are the infinitesimals of the transformations for the variables x, y, t and u, respectively. We shall denote the infinitesimals for u_x, u_y, u_t, u_{xx}, u_{xy}, u_{xt}, ... by ϕ^x, ϕ^y, ϕ^t, ϕ^{xx}, ϕ^{xy}, ϕ^{xt}, ...; for example, we have

$$\begin{aligned} \phi^x &= (\phi-\xi u_x-\eta u_y-\tau u_t)_x+\xi u_{xx}+\eta u_{yx}+\tau u_{tx}\\ &= \phi_x+\phi_u u_x-(\xi_x+\xi_u u_x)u_x-(\eta_x+\eta_u u_x)u_y\\ &\quad -(\tau_x+\tau_u u_x)u_t,\\ \phi^t &= (\phi-\xi u_x-\eta u_y-\tau u_t)_t+\xi u_{xt}+\eta u_{yt}+\tau u_{tt}\\ &= \phi_t+\phi_u u_t-(\xi_t+\xi_u u_t)u_x-(\eta_t+\eta_u u_t)u_y\\ &\quad -(\tau_t+\tau_u u_t)u_t,\\ \phi^{xx} &= (\phi-\xi u_x-\eta u_y-\tau u_t)_{xx}+\xi u_{xxx}+\eta u_{yxx}+\tau u_{txx},\\ \phi^{xt} &= (\phi-\xi u_x-\eta u_y-\tau u_t)_{xt}+\xi u_{xxt}+\eta u_{yxt}+\tau u_{ttx},\\ \phi^{yt} &= (\phi-\xi u_x-\eta u_y-\tau u_t)_{yt}+\xi u_{xyt}+\eta u_{yyt}+\tau u_{tyt},\\ &\ldots \end{aligned} \tag{8}$$

Using these various extensions, the infinitesimals criteria for the invariance of (2) under the group (3) is given by

$$\underline{\mathrm{V}}H\mid_{H=0}=0, \tag{9}$$

where the prolongation of the tangent vector field $\underline{\mathrm{V}}$ is given by

$$\underline{\mathrm{V}} = \xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\tau\frac{\partial}{\partial t}+\phi\frac{\partial}{\partial u}+\phi^x\frac{\partial}{\partial u_x}\\ +\phi^y\frac{\partial}{\partial u_y}+\phi^t\frac{\partial}{\partial u_t}+\phi^{xx}\frac{\partial}{\partial u_{xx}}+\ldots. \tag{10}$$

Here and in the following we denote all $\xi(x,y,t,u)$, $\eta(x,y,t,u)$, $\tau(x,y,t,u)$, $\phi(x,y,t,u)$ by ξ, η, τ, ϕ. The purpose is to solve ξ, η, τ, ϕ by taking (8) and (10) into (9). Then we collect together the coefficients of u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, ... and set all of them to zero. At last we get a system of linear partial differential equations from which we can find ξ, η, τ and ϕ in practice.

Next we will use the above method to find the Lie symmetry group of the nonisospectral KP equation (1). The prolongation of the tangent vector field of (1) is

$$\begin{aligned} \underline{\mathrm{V}} &= \xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\tau\frac{\partial}{\partial t}+\phi\frac{\partial}{\partial u}+\phi^x\frac{\partial}{\partial u_x}\\ &\quad +\phi^y\frac{\partial}{\partial u_y}+\phi^{xx}\frac{\partial}{\partial u_{xx}}+\phi^{xy}\frac{\partial}{\partial u_{xy}}+\phi^{xt}\frac{\partial}{\partial u_{xt}}\\ &\quad +\phi^{yy}\frac{\partial}{\partial u_{yy}}+\phi^{xxxx}\frac{\partial}{\partial u_{xxxx}}. \end{aligned} \tag{11}$$

Then (9) reads [here $H=(4u_t+yu_{xxx}+6yuu_x+2xu_y)_x+3yu_{yy}+4u_y$]:

$$\begin{aligned} &4\phi^{xt}+y\phi^{xxxx}+12yu_x\phi^x+6yu\phi^{xx}+6yu_{xx}\phi\\ &\quad +6\phi^y+2x\phi^{xy}+3y\phi^{yy}+u_{xxxx}\eta+6{u_x}^2\eta\\ &\quad +6uu_{xx}\eta+2u_{xy}\xi+3u_{yy}\eta=0. \end{aligned} \tag{12}$$

Substituting (8) and $u_{xxxx} = -\frac{1}{y}(4u_{xt}+6y{u_x}^2+6yuu_{xx}+6u_y+2xu_{xy}+3yu_{yy})$ for ϕ^x, ϕ^y, ϕ^{xx}, ϕ^{xy}, ϕ^{xt}, ϕ^{yy}, ϕ^{xxxx} in (12), we get a system of linear partial differential equations. We set all the coefficients of u_x, u_y, u_t, u_{xx}, u_{yy}, u_{xt}, ... to zero. Then we get

$$\xi = -2y^{\frac{2}{3}}g_t(t)+y^{\frac{1}{3}}h(t)+xf_t(t)+\frac{1}{3}\frac{xg(t)}{y^{\frac{1}{3}}}\\ -2yf_{tt}(t), \tag{13}$$

$$\eta = 2yf_t(t)+y^{\frac{2}{3}}g(t), \tag{14}$$

$$\tau = f(t), \tag{15}$$

$$\phi = \frac{1}{9}\frac{xh(t)}{y^{\frac{5}{3}}}-2uf_t(t)-\frac{2}{3}\frac{ug(t)}{y^{\frac{1}{3}}}+\frac{2}{3}\frac{h_t(t)}{y^{\frac{2}{3}}}\\ -\frac{2}{9}\frac{xg_t(t)}{y^{\frac{4}{3}}}-\frac{4}{3}f_{ttt}(t)-\frac{4}{3}\frac{g_t(t)}{y^{\frac{1}{3}}}-\frac{1}{27}\frac{x^2g(t)}{y^{\frac{7}{3}}}. \tag{16}$$

So the infinitesimal operator of (1) is

$$
\begin{aligned}
Q &= \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial t} + \phi \frac{\partial}{\partial u} = \\
&\Big(-2y^{\frac{2}{3}} g_t(t) + y^{\frac{1}{3}} h(t) + x f_t(t) + \frac{1}{3}\frac{xg(t)}{y^{\frac{1}{3}}} - 2y f_{tt}(t)\Big)\frac{\partial}{\partial x} \\
&+ \Big(2y f_t(t) + y^{\frac{2}{3}} g(t)\Big)\frac{\partial}{\partial y} + f(t)\frac{\partial}{\partial t} \\
&+ \Big(\frac{1}{9}\frac{xh(t)}{y^{\frac{5}{3}}} - 2u f_t(t) - \frac{2}{3}\frac{ug(t)}{y^{\frac{1}{3}}} + \frac{2}{3}\frac{h_t(t)}{y^{\frac{2}{3}}} - \frac{2}{9}\frac{xg_t(t)}{y^{\frac{4}{3}}} \\
&- \frac{4}{3} f_{ttt}(t) - \frac{4}{3}\frac{g_t(t)}{y^{\frac{1}{3}}} - \frac{1}{27}\frac{x^2 g(t)}{y^{\frac{7}{3}}}\Big)\frac{\partial}{\partial u} = \\
&\Big\{(x f_t(t) - 2y f_{tt}(t))\frac{\partial}{\partial x} + 2y f_t(t)\frac{\partial}{\partial y} + f(t)\frac{\partial}{\partial t} \\
&\quad + \Big(-\frac{4}{3} f_{ttt}(t) - 2u f_t(t)\Big)\frac{\partial}{\partial u}\Big\} \\
&+ \Big\{\Big(\frac{1}{3}\frac{xg(t)}{y^{\frac{1}{3}}} - 2y^{\frac{2}{3}} g_t(t)\Big)\frac{\partial}{\partial x} + y^{\frac{2}{3}} g(t)\frac{\partial}{\partial y} \\
&+ \Big(-\frac{4}{3}\frac{g_{tt}(t)}{y^{\frac{1}{3}}} - \frac{1}{27}\frac{x^2 g(t)}{y^{\frac{7}{3}}} - \frac{2}{9}\frac{xg_t(t)}{y^{\frac{4}{3}}} - \frac{2}{3}\frac{ug(t)}{y^{\frac{1}{3}}}\Big)\frac{\partial}{\partial u}\Big\} \\
&+ \Big\{y^{\frac{1}{3}} h(t)\frac{\partial}{\partial x} + \Big(\frac{2}{3}\frac{h_t(t)}{y^{\frac{2}{3}}} + \frac{1}{9}\frac{xh(t)}{y^{\frac{5}{3}}}\Big)\frac{\partial}{\partial u}\Big\} \\
&= X(f) + Y(g) + Z(h). \qquad (17)
\end{aligned}
$$

Here $f(t)$, $g(t)$ and $h(t)$ are all arbitrary functions of t. Q is a general element of the Lie algebra of (1). The commutation relations for this Lie algebra are easily to obtain:

$$
\begin{aligned}
&[X(f_1), X(f_2)] = X(f_1\dot{f}_2 - \dot{f}_1 f_2), \\
&[X(f), Y(g)] = Y\Big(f\dot{g} - \frac{2}{3}\dot{f}g\Big), \\
&[X(f), Z(h)] = Z\Big(f\dot{h} - \frac{1}{3}\dot{f}h\Big), \\
&[Y(g_1), Y(g_2)] = \frac{2}{3} Z(\dot{g}_1 g_2 - g_1\dot{g}_2), \\
&[Y(g), Z(h)] = 0, \quad [Z(h_1), Z(h_2)] = 0,
\end{aligned}
$$

where the dots indicate derivatives with respect to t. To solve (1), from (17) we can also get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(x', y', t', u') = Q(x', y', t', u'), \\
&(x', y', t', u')|_{\varepsilon=0} = (x, y, t, u),
\end{aligned} \qquad (18)
$$

or

$$
\frac{\mathrm{d}x'}{\xi'} = \frac{\mathrm{d}y'}{\eta'} = \frac{\mathrm{d}t'}{\tau'} = \frac{\mathrm{d}u'}{\phi'} = \mathrm{d}\varepsilon. \qquad (19)
$$

Here ξ', η', τ', ϕ' are obtained by insteading x, y, t, u in ξ, η, τ, ϕ by x', y', t', u'. From (19), (13), (14), (15) and (16), we can get x', y', t', u', but their impressions are too complicated. For simplicity, we only discuss the case where $f(t) = g(t) = 0, h(t) \neq 0$. From (17), we obtain

$$
Q = Z(h(t)) = y^{\frac{1}{3}} h(t)\frac{\partial}{\partial x} + \left(\frac{2}{3}\frac{h_t(t)}{y^{\frac{2}{3}}} + \frac{1}{9}\frac{xh(t)}{y^{\frac{5}{3}}}\right)\frac{\partial}{\partial u}. \qquad (20)
$$

Then we solve the equations

$$
\frac{\mathrm{d}y'}{0} = \frac{\mathrm{d}t'}{0} = \mathrm{d}\varepsilon, \qquad (21)
$$

$$
\frac{\mathrm{d}x'}{\mathrm{d}\varepsilon} = y'^{\frac{1}{3}} h(t'), \qquad (22)
$$

$$
\frac{\mathrm{d}u'}{\mathrm{d}\varepsilon} = \frac{2}{3}\frac{h_{t'}(t')}{y'^{\frac{2}{3}}} + \frac{1}{9}\frac{x'h(t')}{y'^{\frac{5}{3}}}. \qquad (23)
$$

The solution is

$$
\begin{aligned}
u' = &-\frac{1}{18}\frac{h^2(t)}{y^{\frac{4}{3}}}\varepsilon^2 + \frac{1}{3y^{\frac{2}{3}}}\left(2h_t(t) + \frac{1}{3}\frac{h(t)x}{y}\right)\varepsilon \\
&+ u(x - y^{\frac{1}{3}} h(t)\varepsilon, y, t).
\end{aligned} \qquad (24)
$$

That is to say if $u(x,y,t)$ is a solution of (1), u' in (24) is also a solution of (1).

3. Lie Symmetry Group of the Nonisospectral KP Equation Obtained by the Modified C-K Method

In this section we will use another method – a simple direct method – to investigate (1). In [4], Clarkson and Kruskal introduced a direct method to derive symmetry reductions of a nonlinear system without using any group theory. For many types of nonlinear systems the method can be used to find all possible similarity reductions. In [3], Lou and Ma modified the C-K direct method to find the generalized Lie and non-Lie symmetry groups for the well-known KP equation. Here we will use this modified direct method to discuss (1).

At first, we introduce the main idea of this modified C-K direct method [3, 8 – 10]. Given a general partial differential equation (PDE) with the variables x_i and u

$$
F(x_i, u, u_{x_i}, u_{x_i x_j}, \ldots) = 0, \quad i, j = 1, 2, \ldots, n. \qquad (25)
$$

The general form of solutions of (25) reads

$$
\begin{aligned}
&u = W(x_1, x_2, \ldots, x_n, U(X_1, X_2, \ldots, X_n)), \\
&(X_i = X_i(x_1, x_2, \ldots, x_n), i = 1, 2, \ldots, n).
\end{aligned} \qquad (26)
$$

It requires U to satisfy the same PDE (25) under the transformation $(u,x_1,x_2,\ldots,x_n) \longrightarrow (U,X_1,X_2,\ldots,X_n)$.

We can see that it is enough to suppose that the group transformation has some simple forms, say

$$u(x_1,x_2,\ldots,x_n)=\alpha(x_1,x_2,\ldots,x_n) +\beta(x_1,x_2,\ldots,x_n)U(X_1,X_2,\ldots,X_n), \quad (27)$$

instead of (26) for various nonlinear systems especially for those models where the C-K direct reduction method does work.

Next we will apply this method with the proof of the generality of (27) to (1).

At first, we will show that in the following ξ, η, τ are functions of x, y, t and that they are different from ξ, η, τ in Section 2. There are no relations between them. Substituting

$$u=W(x,y,t,U(\xi,\eta,\tau)) \quad (28)$$

into (1) and requiring $U(\xi,\eta,\tau)$ is also a solution of the nonisospectral KP equation but with independent variables (U,ξ,η,τ) [eliminating $U_{\xi\tau}$ and its higher-order derivatives by means of (1)]. Then we get

$$-y^2W_U\xi_x\tau_x{}^3U_{\xi^{10}} +G(x,y,t,U,U_\xi,\ldots,U_{\xi^9},U_\eta,U_\tau,\ldots)=0, \quad (29)$$

where $U_{\xi^n}=\frac{\partial^nU}{\partial\xi^n}$ and G is a complicated $U_{\xi^{10}}$-independent function. Equation (29) is true for an arbitrary solution U only for all coefficients of the polynomials of the derivatives of U being zero. From (29) the coefficient of $U_{\xi^{10}}$ should be zero. So we have

$$y^2W_U\xi_x\tau_x{}^3=0. \quad (30)$$

Obviously, W_U should not be zero, and one can prove that there is no nontrivial solution for $\xi_x=0$; so the only possible case of (30) is

$$\tau_x=0,\ \text{i.e.,}\ \tau=\tau(y,t). \quad (31)$$

Substituting (31) into (29), we get

$$yW_U(3\tau_y{}^2U_{\tau\tau}+\eta_x{}^4U_{\eta^4})+G_1(x,y,t,U,U_\xi,\ldots)=0. \quad (32)$$

Setting the coefficients of $U_{\tau\tau}$ and U_{η^4} in (32) to zero, we have

$$\tau\equiv\tau(t),\quad \eta\equiv\eta(y,t). \quad (33)$$

Using condition (33), (32) is reduced to

$$3yW_{UU}\xi_x{}^4U_{\xi\xi}{}^2+G_2(x,y,t,U,U_\xi,\ldots)=0. \quad (34)$$

Now vanishing the coefficient of $U_{\xi\xi}{}^2$ in (34), we have

$$W_{UU}=0. \quad (35)$$

That is to say we can use

$$u=\alpha(x,y,t)+\beta(x,y,t)U(\xi,\eta,\tau) \quad (36)$$

to substitute the general solution (28). Now, the substitution of (36) with (33) into the nonisospectral KP equation leads to

$$\begin{aligned}&\beta\xi_x(y\xi_x{}^3-\tau_t\eta)U_{\xi\xi\xi\xi}+2\xi_x{}^2y(2\beta_x\xi_x+3\beta\xi_{xx})U_{\xi\xi\xi}\\&+6y(\beta\beta_{xx}+\beta_x{}^2)U^2+6\beta\xi_x(y\beta\xi_x-\tau_t\eta)U_\xi{}^2\\&+\big[6y\beta(4\beta_x\xi_x+y\xi_{xx})U_\xi+6\beta\xi_x(y\beta\xi_x-\tau_t\eta)U_{\xi\xi}\\&+2x\beta_{xy}+y\beta_{xxxx}+6y\alpha\beta_{xx}+6y\beta\alpha_{xx}+3y\beta_{yy}+6\beta_y\\&+4\beta_{xt}+12y\alpha_x\beta_x\big]U+\big(3y\beta\xi_{yy}+4y\beta_x\xi_{xxx}+12y\alpha\beta_x\xi_x\\&+6y\beta_{xx}\xi_{xx}+4\beta_t\xi_x+6y\beta_y\xi_y+12y\beta\alpha_x\xi_x+6\beta\xi_y\\&+4y\xi_x\beta_{xxx}+6y\alpha\beta\xi_{xx}y\beta\xi_{xxxx}+2x\beta_y\xi_x+4\beta_x\xi_t\\&+2x\beta_x\xi_y+4\beta\xi_{xt}+2x\beta\xi_{xt}\big)U_\xi+(4\beta_x\eta_t+6\beta\eta_y\\&+3y\beta\eta_{yy}-6\beta\xi_x\tau_t+6y\beta_y\eta_y+2x\beta_x\eta_y)U_\eta+4\beta_x\tau_tU_\tau\\&+\big(3y\beta\xi_{xx}+6y\alpha\beta\xi_x{}^2+4\beta\xi_x\xi_t+3y\beta\xi_y{}^2+4y\beta\xi_x\xi xxx\\&+12y\beta_x\xi_x\xi_{xx}+6y\beta_{xx}\xi_x{}^2+2x\beta\xi_x\xi_y\big)U_{\xi\xi}+2\beta\big(2\xi_x\eta_t\\&-\xi\xi_x\tau_t+2x\xi_x\eta_y+3y\xi_y\eta_y\big)U_{\xi\eta}+3\beta\big(-\xi_x\tau_t\eta\\&+y\eta_y{}^2\big)U_{\eta\eta}+2x\alpha_{xy}+3y\alpha_{yy}+6y\alpha\alpha_{xx}+y\alpha_{xxxx}\\&+4\alpha_{xt}+6y\alpha_x{}^2+6\alpha_y=0.\end{aligned} \quad (37)$$

Setting the coefficients of the polynomials of U and its derivatives to be zero, (37) can be read as

$$\begin{aligned}&y\xi_x{}^3-\tau_t\eta=0,\quad 2\beta_x\xi_x+3\beta\xi_{xx}=0,\\&\beta\beta_{xx}+\beta_x{}^2=0,\quad 4\beta_x\xi_x+y\xi_{xx}=0,\end{aligned} \quad (38)$$

$$\begin{aligned}&2x\beta_{xy}+y\beta_{xxxx}+6y\alpha\beta_{xx}+6y\beta\alpha_{xx}+3y\beta_{yy}\\&+6\beta_y+4\beta_{xt}+12y\alpha_x\beta_x=0,\end{aligned} \quad (39)$$

$$\begin{aligned}&3y\beta\xi_{yy}+4y\beta_x\xi_{xxx}+12y\alpha\beta_x\xi_x+6y\beta_{xx}\xi_{xx}\\&+4\beta_t\xi_x+6y\beta_y\xi_y+12y\beta\alpha_x\xi_x+6\beta\xi_y\\&+4y\xi_x\beta_{xxx}+6y\alpha\beta\xi_{xx}y\beta\xi_{xxxx}+2x\beta_y\xi_x+4\beta_x\xi_t\\&+2x\beta_x\xi_y+4\beta\xi_{xt}+2x\beta\xi_{xy}=0,\end{aligned} \quad (40)$$

$$4\beta_x\eta_t+6\beta\eta_y+6y\beta_y\eta_y+3y\beta\eta_{yy}-6\beta\xi_x\tau_t+2x\beta_x\eta_y=0,\quad \beta_x\tau_t=0,\quad y\beta\xi_x-\tau_t\eta=0, \tag{41}$$

$$3y\beta\xi_{xx}+6y\alpha\beta{\xi_x}^2+4\beta\xi_x\xi_t+3y\beta{\xi_y}^2+4y\beta\xi_x\xi xxx+12y\beta_x\xi_x\xi_{xx}+6y\beta_{xx}{\xi_x}^2+2x\beta\xi_x\xi_y=0, \tag{42}$$

$$2\xi_x\eta_t-\xi\xi_x\tau_t+x\xi_x\eta_y+3y\xi_y\eta_y=0,\quad -\xi_x\tau_t\eta+y{\eta_y}^2=0, \tag{43}$$

$$2x\alpha_{xy}+3y\alpha_{yy}+6y\alpha\alpha_{xx}+y\alpha_{xxxx}+4\alpha_{xt}+6y{\alpha_x}^2+6\alpha_y=0. \tag{44}$$

The result reads

$$\begin{aligned}\xi=\Big[&{\tau_t}^{\frac{5}{3}}\left(2\delta c_1{\tau_t}^{\frac{2}{3}}y^{\frac{2}{3}}\eta_0+{c_1}^2{\eta_0}^2y^{\frac{1}{3}}+y{\tau_t}^{\frac{4}{3}}\right)x-2y^2{\tau_t}^2\tau_{tt}-2\delta c_1{\tau_t}^{\frac{4}{3}}\left(3\tau_t\eta_{0t}+2\tau_{tt}\eta_0\right)y^{\frac{5}{3}}\\&-{\tau_t}^{\frac{2}{3}}\left(2{c_1}^2{\eta_0}^2\tau_{tt}+6{c_1}^2\tau_t\eta_0\eta_{0t}-3\delta{\tau_t}^{\frac{2}{3}}\xi_0\right)y^{\frac{4}{3}}+6\tau_t\eta_0\left(\delta\eta_0\eta_{0t}+c_1{\tau_t}^{\frac{1}{3}}\xi_0\right)y\\&+3{\tau_t}^{\frac{1}{3}}{\eta_0}^2\left(2c_1\eta_0\eta_{0t}+\delta{c_1}^2{\tau_t}^{\frac{1}{3}}\xi_0\right)y^{\frac{2}{3}}\Big]\Big/\left({\tau_t}^{\frac{4}{3}}y^{\frac{2}{3}}\left(\delta y^{\frac{1}{3}}{\tau_t}^{\frac{2}{3}}+c_1\eta_0\right)\right),\end{aligned} \tag{45}$$

$$\eta=\left(\delta{c_1}^2y^{\frac{1}{3}}{\tau_t}^{\frac{2}{3}}+\eta_0\right)^3,\quad \beta={\tau_t}^{\frac{2}{3}}\left(\delta{\tau_t}^{\frac{2}{3}}+\frac{c_1\eta_0}{y^{\frac{1}{3}}}\right)^2, \tag{46}$$

$$\begin{aligned}\alpha=\Big\{&{\tau_t}^{\frac{7}{3}}\eta_0\left(y^{\frac{2}{3}}{\eta_0}^5+6\delta{c_1}^2{\tau_t}^{\frac{2}{3}}{\eta_0}^4y+9{c_1}^2{\tau_t}^{\frac{8}{3}}y^2\eta_0+16\delta{\tau_t}^2y^{\frac{5}{3}}{\eta_0}^2+2\delta c_1{\tau_t}^{\frac{10}{3}}y^{\frac{7}{3}}+14c_1{\tau_t}^{\frac{4}{3}}y^{\frac{4}{3}}\right)x^2\\&+\Big[4c_1\delta{\tau_t}^{\frac{14}{3}}\left(3\tau_t\eta_{0t}-2\tau_{tt}\eta_0\right)y^{\frac{10}{3}}+6{\tau_t}^4\left(-\delta{\tau_t}^{\frac{4}{3}}\xi_0+6{c_1}^2\tau_t\eta_0\eta_{0t}-6{c_1}^2\tau_{tt}{\eta_0}^2\right)y^3\\&+8{\tau_t}^{\frac{10}{3}}\eta_0\left(-8\delta\tau_{tt}{\eta_0}^2+3\delta\tau_t\eta_{0t}\eta_0-3c_1{\tau_t}^{\frac{4}{3}}\xi_0\right)y^{\frac{8}{3}}\\&-4{\tau_t}^{\frac{8}{3}}{\eta_0}^2\left(14c_1{\eta_0}^2\tau_{tt}+9\delta{c_1}^2{\tau_t}^{\frac{4}{3}}\xi_0+6c_1\tau_t\eta_0\eta_{0t}\right)y^{\frac{7}{3}}\\&-4{\tau_t}^2{\eta_0}^3\left(6\delta{c_1}^2{\eta_0}^2\tau_{tt}+9\delta{c_1}^2\tau_t\eta_0\eta_{0t}-6{\tau_t}^{\frac{4}{3}}\xi_0\right)y^2\\&-2{\tau_t}^{\frac{4}{3}}{\eta_0}^4\left(3\delta c_1{\tau_t}^{\frac{4}{3}}\xi_0+6\tau_t\eta_0\eta_{0t}+2{\eta_0}^2\tau_{tt}\right)y^{\frac{5}{3}}\Big]x+12{\tau_t}^{\frac{13}{3}}\left(2\tau_t\tau_{ttt}-3{\tau_{tt}}^2\right)y^{\frac{14}{3}}\\&+8\delta c_1{\tau_t}^{\frac{11}{3}}\left(18\tau_t\tau_{ttt}\eta_0-26\tau_{tt}\eta_0-12\tau_t\eta_{0t}\tau_{tt}+9{\tau_t}^2\eta_{0tt}\right)y^{\frac{13}{3}}\\&+36{\tau_t}^3\Big(\delta{\tau_t}^{\frac{4}{3}}\tau_{tt}\xi_0-12{c_1}^2\tau_t\tau_{tt}\eta_{0t}\eta_0+10{c_1}^2{\tau_t}^2\eta_{0tt}\eta_0+10{c_1}^2\tau_t\tau_{ttt}{\eta_0}^2\\&\qquad-14{c_1}^2{\tau_{tt}}^2{\eta_0}^2-\delta{\tau_t}^{\frac{7}{3}}\xi_{0t}-4\delta{\tau_t}^2\eta_0t^2\Big)y^4\\&+{\tau_t}^{\frac{7}{3}}\Big(-696\delta\tau_t\tau_{tt}\eta_{0t}{\eta_0}^2+192c_1{\tau_t}^{\frac{4}{3}}\tau_{tt}\eta_0\xi_0+480\delta\tau_t{\eta_0}^3\tau_{ttt}-216c_1{\tau_t}^{\frac{7}{3}}\eta_0\xi_{0t}\\&\qquad-656\delta{\eta_0}^3{\tau_{tt}}^2+648\delta{\tau_t}^2{\eta_0}^2\eta_{0tt}-720\delta{\tau_t}^2{\eta_{0t}}^2\eta_0+36c_1{\tau_t}^{\frac{7}{3}}\eta_{0t}\xi_0\Big)y^{\frac{11}{3}}\\&+{\tau_t}^{\frac{5}{3}}\Big(-484c_1{\eta_0}^4{\tau_{tt}}^2+360c_1\tau_t{\eta_0}^4\tau_{ttt}+108\delta{c_1}^2{\tau_t}^{\frac{7}{3}}\eta_{0t}\xi_0\eta_0-540\delta{c_1}^2{\tau_t}^{\frac{7}{3}}{\eta_0}^2\xi_{0t}-9{\tau_t}^{\frac{8}{3}}{\xi_0}^2\\&\qquad+360c_1{\tau_t}^2{\eta_0}^3\eta_{0tt}-1584c_1{\tau_t}^2{\eta_0}^2{\eta_{0t}}^2+432\delta{c_1}^2{\tau_t}^{\frac{4}{3}}\tau_{tt}{\eta_0}^2\xi_0-312c_1\tau_t{\eta_0}^3\eta_{0t}\tau_{tt}\Big)y^{\frac{10}{3}}\\&+\tau_t\eta_0\Big(144\delta{c_1}^2\tau_t{\eta_0}^4\tau_{ttt}-36\delta c_1{\tau_t}^{\frac{8}{3}}{\xi_0}^2-720{\tau_t}^{\frac{7}{3}}{\eta_0}^2\xi_{0t}-360\delta a{c_1}^2{\tau_t}^2{\eta_0}^3\eta_{0tt}\\&\qquad+432\delta{c_1}^2\tau_t\tau_{ttt}{\eta_0}^3\eta_{0t}+72{\tau_t}^{\frac{7}{3}}\eta_{0t}\xi_0\eta_0-2016\delta{c_1}^2{\tau_t}^2{\eta_0}^2{\eta_{0t}}^2-192\delta{c_1}^2{\eta_0}^4{\tau_{tt}}^2\\&\qquad+528{\tau_t}^{\frac{4}{3}}{\eta_0}^2\tau_{tt}\xi_0\Big)y^3\\&+{\tau_t}^{\frac{1}{3}}{\eta_0}^2\Big(-32{\eta_0}^4{\tau_{tt}}^2+372\delta c_1{\tau_t}^{\frac{4}{3}}{\eta_0}^2\tau_{tt}\xi_0-54{c_1}^2{\tau_t}^{\frac{8}{3}}{\xi_0}^2-540\delta c_1{\tau_t}^{\frac{7}{3}}{\eta_0}^2\xi_{0t}\\&\qquad+672\tau_t{\eta_0}^3\tau_{tt}\eta_{0t}-72\delta c_1{\tau_t}^{\frac{7}{3}}\eta_0\eta_{0t}\xi_0-1584{\tau_t}^2{\eta_0}^2{\eta_{0t}}^2+24\tau_t{\eta_0}^4\tau_{ttt}-648{\tau_t}^2{\eta_0}^3\eta_{0tt}\Big)y^{\frac{8}{3}}\\&+36{\tau_t}^{\frac{2}{3}}{\eta_0}^3\Big(4{c_1}^2{\tau_t}^{\frac{1}{3}}{\eta_0}^2\tau_{tt}\xi_0-3{c_1}^2{\tau_t}^{\frac{2}{3}}\xi_0\eta_0\eta_{0t}-\delta{\tau_t}^{\frac{5}{3}}{\xi_0}^2+10\delta c_1{\eta_0}^3\tau_{tt}\eta_{0t}\\&\qquad-6{c_1}^2{\tau_t}^{\frac{2}{3}}{\eta_0}^2\xi_{0t}-10\delta c_1\tau_t{\eta_0}^3\eta_{0tt}-20\delta c_1\tau_t{\eta_0}^2{\eta_{0t}}^2\Big)y^{\frac{7}{3}}\end{aligned}$$

$$+\eta_0^4\left(-36\delta\tau_t^{\frac{4}{3}}\eta_0^2\xi_{0t}+24\delta\tau_t^{\frac{1}{3}}\eta_0^2\tau_{tt}\xi_0-144c_1^2\tau_t\eta_0^2\eta_{0t}^2+72c_1^2\eta_0^3\tau_{tt}\eta_{0t}\right.$$
$$\left.-36\delta\tau_t^{\frac{4}{3}}\eta_0\xi_0\eta_{0t}-9c_1\tau_t^{\frac{5}{3}}\xi_0^3-72c_1^2\tau_t\eta_0^3\eta_{tt}\right)y^2\Big\}\Big/$$
$$\left\{\tau_t^{\frac{7}{3}}y^{\frac{8}{3}}\left(2\delta c_1y^{\frac{1}{3}}\tau_t^{\frac{2}{3}}\eta_0+c_1^2\eta_0^2+\tau_t^{\frac{4}{3}}y^{\frac{2}{3}}\right)^2\left(\delta y^{\frac{1}{3}}\tau_t^{\frac{2}{3}}+c_1\eta_0\right)^2\right\}. \tag{47}$$

Here $\xi_0\equiv\xi_0(t)$, $\eta_0\equiv\eta_0(t)$, $\tau\equiv\tau(t)$ are arbitrary functions of time t, while the constants δ and c_1 possess discrete values determined by

$$\delta=1,\quad -1,$$

$$c_1=1,\quad \frac{-1+\mathrm{i}\sqrt{3}}{2},\quad \frac{-1-\mathrm{i}\sqrt{3}}{2}\quad (\mathrm{i}=\sqrt{-1}).$$

At last, we get, if $U=U(x,y,t)$ is a solution of the nonisospectral KP equation

$$u=\alpha+\tau_t^{\frac{2}{3}}\left(\delta\tau_t^{\frac{2}{3}}+\frac{c_1\eta_0}{y^{\frac{1}{3}}}\right)^2U(\xi,\eta,\tau) \tag{48}$$

with (45), (46) and (47), where ξ_0, η_0, τ are arbitrary functions of t. From (48) with (45), (46) and (47) we know that for the real nonisospectral KP equation the symmetry group is divided into two sections: the Lie point symmetry group which corresponds to

$$\delta=1,\quad c_1=1,$$

and a coset of the Lie group which is related to

$$\delta=-1,\quad c_1=1.$$

The coset is equivalent to the reflected transformation of x and y, i. e., $\{y\to-y,t\to-t\}$ accompanied by the usual Lie point symmetry transformation.

We denote by $\mathcal{S}$ the Lie point symmetry group of the real nonisospectral KP equation (NKP), by σ the reflection of $\{y\to-y,t\to-t\}$, by I the identity transformation and by $\mathcal{C}_2\equiv\{I,\sigma\}$ the discrete reflection group. Then the full Lie symmetry group $\mathcal{G}_{\mathrm{RNKP}}$ of the real nonisospectral KP equation can be expressed as

$$\mathcal{G}_{\mathrm{RNKP}}=\mathcal{C}_2\otimes\mathcal{S}.$$

For the complex nonisospectral KP equation, the symmetry group is divided into six sectors which correspond to

$$\delta=1,\quad c_1=1,$$
$$\delta=1,\quad c_1=\tfrac{-1+\sqrt{3}\mathrm{i}}{2},$$
$$\delta=1,\quad c_1=\tfrac{-1-\sqrt{3}\mathrm{i}}{2},$$
$$\delta=-1,\quad c_1=1,$$
$$\delta=-1,\quad c_1=\tfrac{-1+\sqrt{3}\mathrm{i}}{2},$$
$$\delta=-1,\quad c_1=\tfrac{-1-\sqrt{3}\mathrm{i}}{2}.$$

That is to say, the full symmetry group, $\mathcal{G}_{\mathrm{CNKP}}$, expressed by (48) with (45), (46) and (47) for the complex nonisospectral KP equation, is the product of the usual Lie point symmetry group $\mathcal{S}$ ($\delta=1,c_1=1$) and the discrete group $\mathcal{D}_3$, i. e.,

$$\mathcal{G}_{\mathrm{CNKP}}=\mathcal{D}_3\otimes\mathcal{S},$$
$$\mathcal{D}_3\equiv\{I,\sigma^{yt},R_1,R_2,\sigma^{yt}R_1,\sigma^{yt}R_2\},$$

where I is the identity transformation and

$$\sigma^{yt}:\{y,t\}\to\{-y,-t\},$$

$$R_1:u(x,y,t)\to$$
$$-\frac{1+\sqrt{3}\mathrm{i}}{2}u\left(\frac{-1+\sqrt{3}\mathrm{i}}{2}x,-\frac{1+\sqrt{3}\mathrm{i}}{2}y,\frac{-1+\sqrt{3}\mathrm{i}}{2}t\right),$$

$$R_2:u(x,y,t)\to$$
$$\frac{-1+\sqrt{3}\mathrm{i}}{2}u\left(-\frac{1+\sqrt{3}\mathrm{i}}{2}x,\frac{-1+\sqrt{3}\mathrm{i}}{2}y,-\frac{1+\sqrt{3}\mathrm{i}}{2}t\right).$$

We will show that the Lie symmetry group obtained via the traditional Lie approach is only a special case of the symmetry groups obtained by the modified C-K method. When $\delta=1$, $c_1=1$, we can find the Lie point symmetry group from (48) with (45), (46) and (47) and we can see its equivalence with the result obtained in Section 2.

We set

$$\tau(t)=t+\varepsilon f(t),\ \xi_0(t)=\frac{1}{3}\varepsilon h(t),\ \eta_0(t)=\frac{1}{3}\varepsilon g(t), \tag{49}$$

with an infinitesimal parameter ε. Then (48) with (45), (46) and (47) with $\delta=1$, $c_1=1$ can be written as

$$u=U+\varepsilon\sigma(U)+O(\varepsilon^2),$$

$$\sigma(U)=\left\{xf_t(t)+\frac{1}{3}\frac{xg(t)}{y^{\frac{1}{3}}}-2y^{\frac{2}{3}}g_t(t)-2yf_{tt}(t)+y^{\frac{1}{3}}h(t)\right\}U_x+\left\{2yf_t(t)+g(t)y^{\frac{2}{3}}\right\}U_y+f(t)U_t$$
$$+\left\{\left(\frac{2}{3}\frac{g(t)}{y^{\frac{1}{3}}}+2f_t(t)\right)U+\frac{4}{3}f_{ttt}(t)+\frac{1}{27}\frac{x^2g(t)}{y^{\frac{7}{3}}}+\frac{4}{3}\frac{g_{tt}(t)}{y^{\frac{1}{3}}}+\frac{2}{9}\frac{xg_t(t)}{y^{\frac{4}{3}}}-\frac{1}{9}\frac{xh(t)}{y^{\frac{5}{3}}}-\frac{2}{3}\frac{h_t(t)}{y^{\frac{2}{3}}}\right\}. \quad (50)$$

The equivalent vector expression of the above symmetry is

$$V=\left\{(xf_t(t)-2yf_{tt}(t))\frac{\partial}{\partial x}+2yf_t(t)\frac{\partial}{\partial y}+f(t)\frac{\partial}{\partial t}+(-2Uf_t(t)-\frac{4}{3}f_{ttt})\frac{\partial}{\partial U}\right\}$$
$$+\left\{\left(-2y^{\frac{2}{3}}g_t(t)+\frac{1}{3}\frac{xg(t)}{y^{\frac{1}{3}}}\right)\frac{\partial}{\partial x}+y^{\frac{2}{3}}g(t)\frac{\partial}{\partial y}+\left(-\frac{2}{3}\frac{Ug(t)}{y^{\frac{1}{3}}}-\frac{4}{3}\frac{g_{tt}}{y^{\frac{1}{3}}}-\frac{1}{27}\frac{x^2g(t)}{y^{\frac{7}{3}}}-\frac{2}{9}\frac{xg_t(t)}{y^{\frac{4}{3}}}\right)\frac{\partial}{\partial U}\right\} \quad (51)$$
$$+\left\{y^{\frac{1}{3}}h(t)\frac{\partial}{\partial x}+\left(\frac{2}{3}\frac{h_t(t)}{y^{\frac{2}{3}}}+\frac{1}{9}\frac{xh(t)}{y^{\frac{5}{3}}}\right)\frac{\partial}{\partial U}\right\}=X(f(t))+Y(g(t))+Z(h(t)).$$

We can see that (51) is exactly the same as (17) which we have obtained by the standard Lie approach in Section 2.

Remark: In [3], Lou and Ma have obtained the full symmetry group of the complex KP equation. It can be divided into six sectors. In our result, the full symmetry group of the complex nonisospectral KP equation is also be gotten, similarly to the KP equation, and it also has six sectors. It is very interesting that the full Lie symmetry groups of the isospectral and nonisospectral KP equation have the similar algebraic structure.

4. Conclusions

Based on two methods: the classical symmetry method and a simple direct method, two Lie symmetry groups of a nonisospectral KP equation have been constructed. It has been shown that the Lie symmetry group obtained via the traditional Lie approach is only a special case of the full symmetry groups obtained by the modified C-K method. Using the modified C-K method, we can also get more solutions of the equation from the old ones. At the same time, we showed that the full symmetry groups of the isospectral and nonisospectral KP equation have the same algebraic structure. However, from our result, it was shown that the nonisospectral problem is much more complicated than the isospectral one. Its calculation is more complex and corresponding reduction is more difficult than the isospectral one. That might be because of its inherent variable spectral parameter which also makes the equation more valuable and general.

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